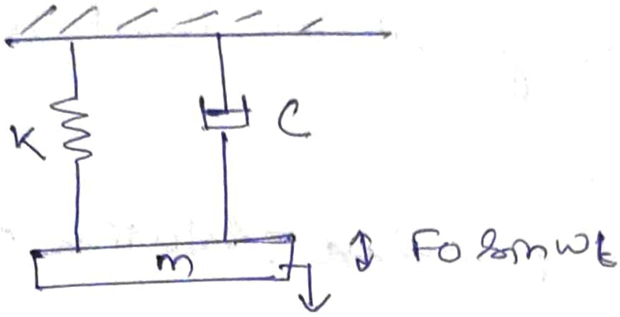
Forced vibration of SDOF System

If the external force or an imposed displacement excitation is applied to a System during vibration the mechanical System is said to ~~be~~ undergo forced vibration.

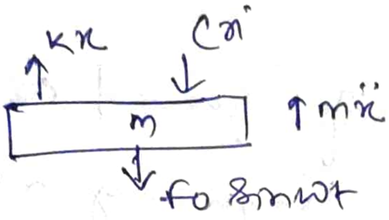
- The applied force may be harmonic, non-harmonic, periodic, non periodic or random in nature.
- When the System is subjected to forced harmonic excitation its vibration response takes place at the same frequency as that of the excitation.
- The common sources of harmonic excitation are
  - Rotating unbalance
  - Reciprocating unbalance
  - M/C motion itself (Seismic)

Damped forced vibration



Here we will discuss the behaviour of the system acted upon by harmonic force.

$F = F_0 \sin\omega t$



(FBD)

Applying D'Alembert principle.

$$m\ddot{x} + c\dot{x} + kx - F_0 \sin\omega t = 0$$

$$\text{or } m\ddot{x} + c\dot{x} + kx = F_0 \sin\omega t \quad \text{---} (1)$$

where eqn (1) is the second order linear differential equation with constant coefficient.

The general solution for the non-homogeneous ordinary differential equation contains two the complementary part and particular parts. i.e. the solution.

$$x(t) = x_c(t) + x_p(t)$$

The complementary part is the solution of homogeneous equation.

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\text{solution is} = x(t) = e^{-\xi\omega_n t}(A\sin\omega_d t + B\cos\omega_d t)$$

where A & B are the constants found out for the initial condition.

For particular Solution -

The Steady State oscillation of the same frequency 'ω' as that of the excitation.

Let's assume the particular Solution x_p

$$x(t) = X \sin(\omega t - \phi)$$

where X = Amplitude of forced oscillation

ω = excitation frequency in rad/sec.

ϕ = phase of the displacement w.r.t exciting force.

Spring force $= f_s = k\, x(t)$

$$= k X \sin(\omega t - \phi)$$

Damping force $= F_d = c\dot{x}$

$$= c \omega X \cos(\omega t - \phi)$$

$$= c \omega X \sin(\omega t - \phi + \pi/2)$$

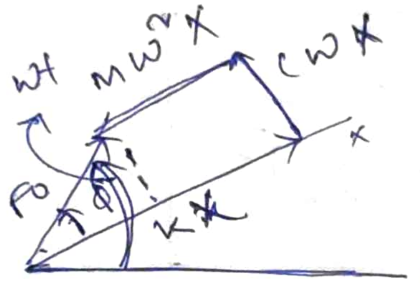
(phase lead)

ie the damping force leads the spring force by $\pi/2$.

Inertia force $= m\ddot{x} = -m\omega^2 X \sin(\omega t - \phi)$

$$= +m\omega^2 X \sin(\omega t - \phi + \pi)$$

Hence the inertia force leads to spring force by π



(vector relationship for forced vibration with damping.)

From the above figure, excitation force Fo is

$$F_0 = \sqrt{(kX - m\omega^2 X)^2 + (c\omega X)^2}$$

$$= X\sqrt{(k - m\omega^2) + (c\omega)^2} \quad \text{——— (2)}$$

Amplitude $X = \dfrac{F_0}{\sqrt{(k - m\omega^2) + (c\omega)^2}}$

phase angle $\phi = \tan^{-1}\dfrac{c\omega}{k - m\omega^2}$

From ~~system at~~ equation (2)

$$X = \frac{F_0}{\sqrt{(k - m\omega^2) + (c\omega)^2}}$$

Deviding Nr and Dr by k, on right side

$$X = \frac{F_0/k}{\sqrt{\left(\frac{k}{k} - \frac{m\omega^2}{k}\right)^2 + \left(\frac{c\omega}{k}\right)^2}} = \frac{F_0/k}{\sqrt{\left(1 - \frac{m\omega^2}{k}\right)^2 + \left(\frac{c\omega}{k}\right)^2}}$$

$$= \frac{F_0/k}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(\frac{c\omega}{k}\right)^2}}$$

& $\dfrac{c\omega}{k} = \dfrac{c}{c_c} \times \dfrac{c_c\omega}{k}$

$= \xi \cdot \dfrac{2m\omega_n\omega}{k}$

$= 2\xi\dfrac{\omega}{\omega_n}$

or $$\boxed{X = \frac{F_0/K}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}}$$

phase angle = $\boxed{\phi = \tan^{-1} \frac{2\xi(\omega/\omega_n)}{1-(\omega/\omega_n)^2}}$

Putting $\left(\frac{\omega}{\omega_n} = \frac{\text{excitation frequency}}{\text{natural frequency}} = r\right)$

the above equation reduced to

$$\boxed{X = \frac{F_0/K}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}}$$

and phase angle $\boxed{\phi = \tan^{-1} \frac{2\xi r}{1-r^2}}$

Magnification factor -(Mf) = It is defined as the ratio of amplitude of steady state vibration to static deflection of mass.

$$Mf = \frac{\text{Amplitude of steady state}}{\text{static deflection}} = \frac{X}{F_0/K} = \frac{KX}{F_0}$$

$$= \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

At resonance, i.e when the frequency of excitation equal to the natural frequency of vibrating system

$$\frac{\omega}{\omega_n} = 1 \quad \text{or} \quad \omega_n = \omega$$

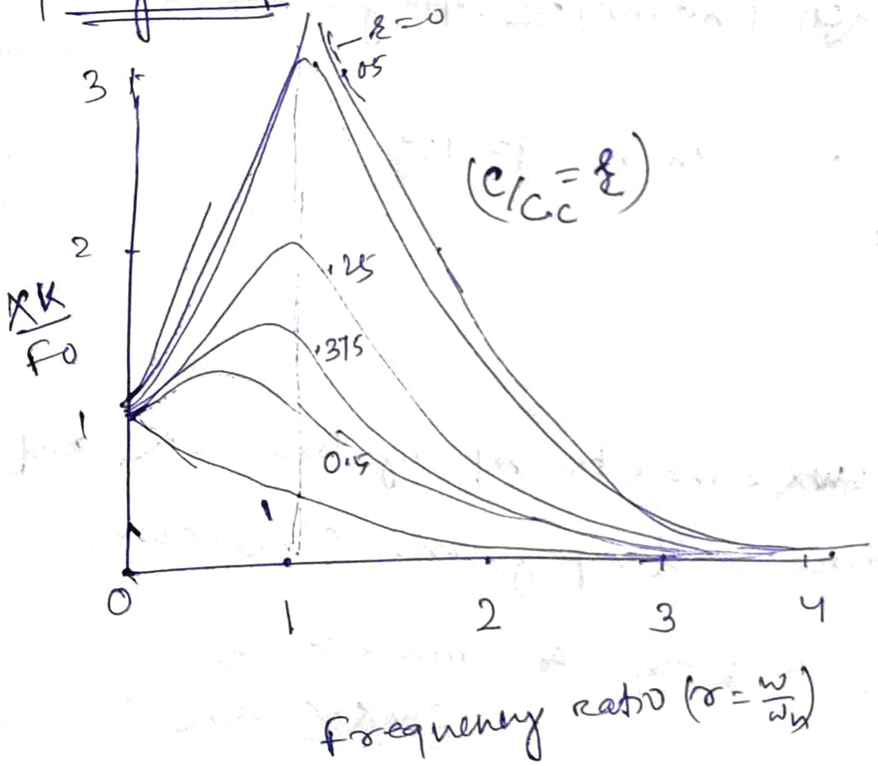
$$M_f = \frac{kx}{F_0} = \frac{1}{2\xi}$$

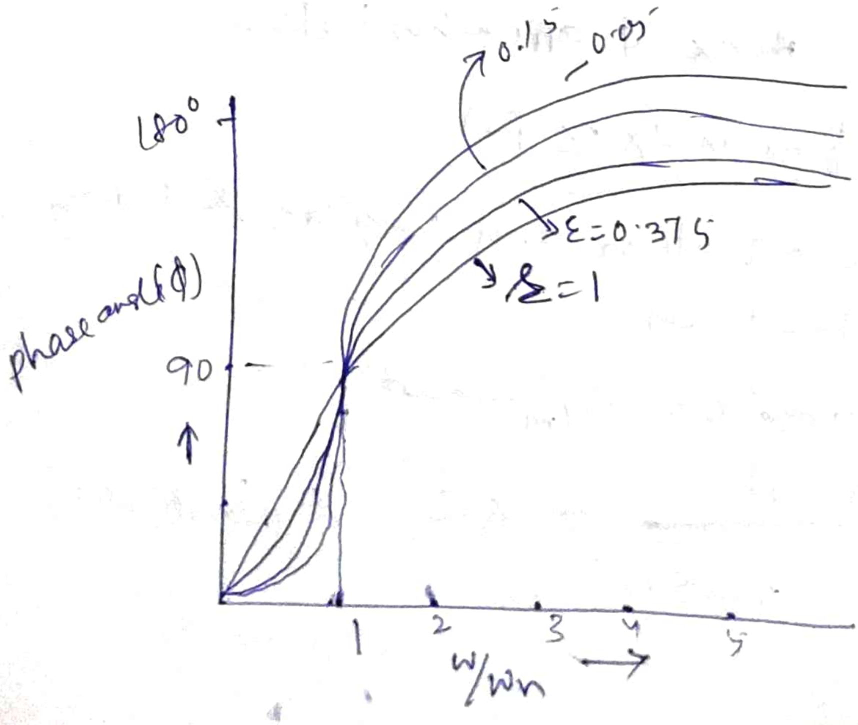
$$\boxed{X_{Res} = \frac{f_0}{k} \cdot \frac{1}{2\xi}}$$

← Amplitude

M to be maximum — $\frac{dM}{dr} = 0$, then $\boxed{M_{max} = \frac{1}{2\xi\sqrt{1-\xi^2}}}$

frequency response





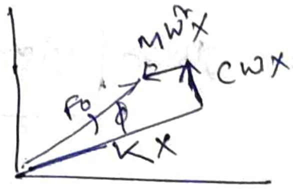
variation of frequency ratio - ω/ω_n

Case - 1

If $r \ll 1$, i.e. $\frac{\omega}{\omega_n} \ll 1$ or ω is very small

the phase angle is very small

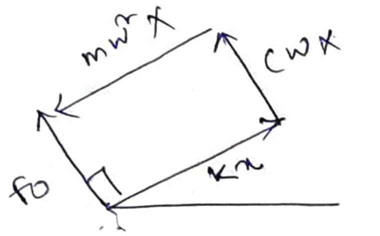
as $c\omega X$ and $m\omega^2 X$ are small and very small



Here f_0 (excitation force) $\simeq$ spring force (kx)

Case - 2

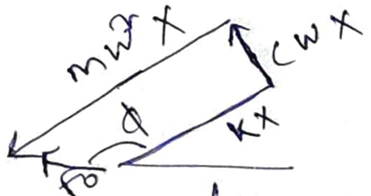
when $r = 1$, $\omega = \omega_n$, $\phi = 90°$



Inertia force $m\omega^2 x$ equal to spring force kx and excitation force = damping force ($f_0 = c\omega x$)

Case - 3 when $r \gg 1$, $\omega \gg \omega_n$ or $m\omega^2 X \gg kx$

$c\omega X > kx$



Here ϕ approaches to $180°$

(Inertia force $m\omega^2 x \simeq f_0$)

* The differential equation and complete solution including transient term as

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = f_0/m \sin\omega t$$

$$x(t) = f_0/k \frac{\sin(\omega t - \phi)}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} + X e^{-\xi\omega_n t} \sin(\omega_d t + \phi)$$

Q In a vibratory system of mass 3 kg vibrates in a viscous medium. A harmonic force of 30N acts on the system and cause a resonance amplitude of 15 mm with a period of 0.25 sec. Find the damping coefficient. If the frequency of the exciting ~~force~~ frequency is changed to 5 Hz, Determine the increase in amplitude of forced vibration upon the removal of the damper.

Soln $m = 3\,kg$ $F_0 = 30N$ $X = 15\,mm$, $T = 0.25$ sec.

$$\text{As } \omega = \frac{2\pi}{T} = \frac{2\pi}{0.25} = 25.13 \text{ rad/sec}$$

(i) At resonance $\omega = \omega_n = \sqrt{K/M} = \sqrt{K/3}$

$$25.13 = \sqrt{K/m}, \quad K = 1894.55\ N/m$$

$$\text{As } X = \frac{F_0/K}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$X_{res} = \frac{F_0/K}{\frac{1}{2\xi}}, \quad 0.015 = \frac{(30/1895.55)}{2\xi}$$

$$\text{or } \xi = 0.528$$

$$\text{Damping coefficient} = C = 2\xi\sqrt{KM} = 79.614\text{-s/m}$$

(ii) $f = 5\,Hz$, $\omega = 2\pi f = 2\pi \times 5 = 31.41$ rad/sec.

$$(X_1)\text{ with damper} = \frac{(30/1894.55)}{\sqrt{\left[1-\left(\frac{31.41}{25.13}\right)^2\right]^2 - \left(2\times 0.528 \times \frac{31.41}{25.13}\right)^2}} = 0.011\,m$$

$$(X_2)\text{ without damper} = \frac{(30/1894.55)}{\sqrt{\left[1-\left(\frac{31.41}{25.13}\right)^2\right]^2}} = 0.028\,M$$

$$\boxed{\text{Increase} = X_2 - X_1 = 0.017}$$

Q. A vibrating system having mass of 20kg. It is suspended from a spring which deflects 10mm under weight of mass. Determine the natural frequency of the free vibration. What is the viscous damping force need to get the aperiodic motion at speed 1mm/s.

If, when damped to this extent, an excitation force of maximum value of 110N and vibrating at 5Hz is made to act on the body. Calculate the amplitude of the ultimate motion.

Soln

$m = 20kg$ $\quad \delta_{st} = 10mm$ $\quad F_0 = 110N$ $\quad f = 5Hz$

$\dot{x} = 1mm/s = 0.001m/s$

$$k = \frac{mg}{\delta_{st}} = \frac{20 \times 9.81}{0.01} = 19620 N/m$$

$$w_n = \sqrt{k/m} = \sqrt{\frac{19620}{20}} = 31.32 \text{ rad/sec}.$$

Again $w_n = 2\pi f_n$, $\quad f_n = \frac{w_n}{2\pi} = \frac{31.32}{2\pi} = 4.918 Hz$

The motion is aperiodic, when $w_d = 0$ or critically damped.

$$C = C_c = 2m w_n = 2 \times 20 \times 31.32$$

$$= 1252.8 \text{ N-S/m}$$

The force needed $= F = C\dot{x}$

$$= 1252.8 \times 0.001 = 1.2528\,N.$$

$$\omega = 2\pi f = 2\pi \times 5 = 10\pi$$

Amplitude

$$x = \frac{F_0}{\sqrt{(c\omega)^2 + (k - m\omega^2)^2}}$$

$$= \frac{110}{\sqrt{(1252.8 \times 10\pi)^2 + (19620 - 20 \times (10\pi)^2)^2}}$$

$$= 2.79 \times 10^{-3} = 2.79\,mm.$$

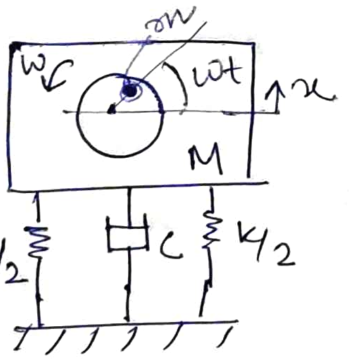
Rotating unbalance

Unbalance in rotating M/C is a common sources of vibration excitation. Any M/C which has a rotor as its working part is called rotating M/C. ex- turbine electric motor.

The problem of unbalance comes in rotating when the CG of rotor does not coincide with axis of rotation.

The unbalance is represented by an eccentric mass 'm' with eccentricity which is rotating with angular velocity 'ω'.

'x' be the displacement of non-rotating mass (M−m) from equilibrium position.



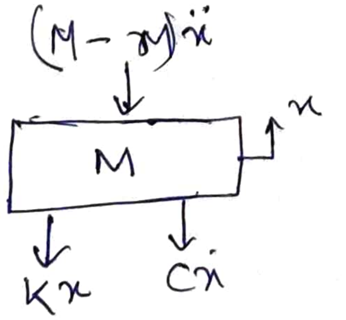
Let m = rotating mass

e = eccentricity of rotating mass

M = Total vibrating mass including 'm'

x = displacement

M−m = non rotating mass



co-ordinate of m = $x + e\sin\omega t$

x ↓ dynamic, $e\sin\omega t$ ↓ Rotating

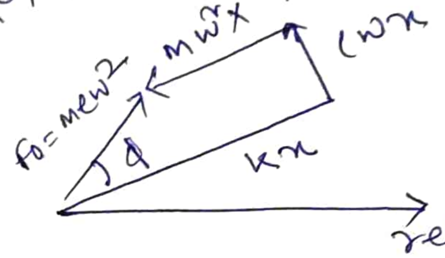
The equation of motion is then

$$\underbrace{(M-m)\ddot{x}}_{\text{Inertia force of non-rotating mass}} + m\frac{d^2}{dt^2}\underbrace{(x+e\sin\omega t)}_{\text{Inertia force of non rotating mass}} + c\dot{x} + kx = 0 \quad \text{---} (1)$$

Rearranging

$$M\ddot{x} + c\dot{x} + kx = (me\omega^2)\sin\omega t$$
$$\approx f_0 \sin\omega t$$

Rotation vector representation



then $$X = \frac{me\omega^2}{(k-M\omega^2)^2+(c\omega)^2} \quad \text{---} (2)$$

Diving Nr and Dr by k, and written $\boxed{\omega_n = \sqrt{k/M}}$

we get

$$X = \frac{me\omega^2/k}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$$, where $= r = \frac{\omega}{\omega_n}$ ---(3)

$$\phi = \tan^{-1}\frac{c\omega}{(k-M\omega^2)}$$

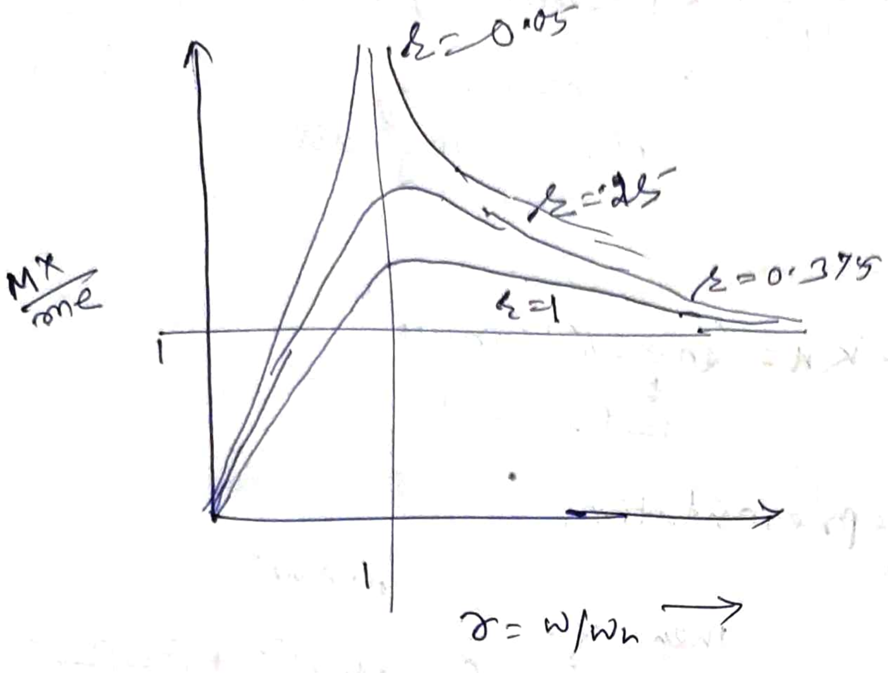
from eqn (3)

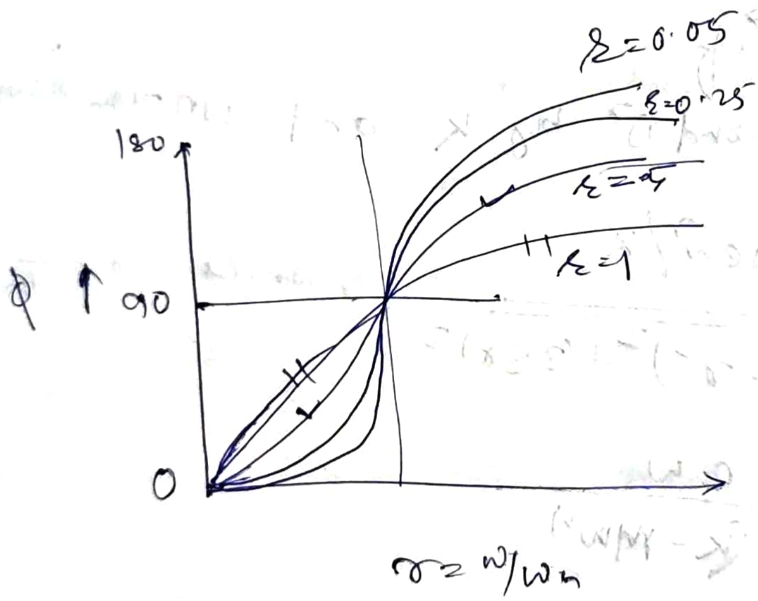
$$\frac{Xk}{me\omega^2}\cdot\frac{M}{M} = \frac{MX}{me(\omega/\omega_n)^2} = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+(2\xi\,\omega/\omega_n)^2}}$$

$$\boxed{\frac{MX}{me} = \frac{(\omega/\omega_n)^2}{\sqrt{[1-(\omega/\omega_n)^2]^2+(2\xi\,\omega/\omega_n)^2}}}$$

$$\tan\phi = \frac{2\xi(\omega/\omega_n)}{1-(\omega/\omega_n)^2}$$

Frequency response





Q. A counter rotating eccentric mass is used to determine the vibrational characteristics of a structure of mass 180kg. At a speed of 900 rpm a stroboscope shows that the eccentric makes to be at the top at the instant the structure is moving upward through its static equilibrium position and the corresponding amplitude 20mm. If the unbalanced of each wheel of exciter is 0.092 kg-m. Determine.

ⓐ The natural frequency of the structure

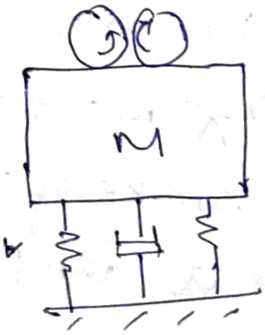
ⓑ damping factor of the structure

ⓒ Amplitude at 1200 rpm

ⓓ The angular position of the eccentrics at the instant, the structure moving upward through its equilibrium position.

Solⁿ $M = 180\,kg$ $\quad N = 900\,rpm \quad X = 20$

$me = 0.092$ (of each) (kg-m)



$w = \frac{2\pi N}{60} = 94.25\,rad/sec.$

ⓐ As here $w = w_n$, $r = 1$

at $r = 1$, $w = w_n = 94.25$ $\quad f_n = \frac{94.25}{2\pi} = 15\,Hz$

ⓑ $\frac{MX}{me} = \frac{1}{2\xi}$

$\xi = \frac{me}{2 \times MX} = \frac{0.092 \times 2}{2 \times 180 \times 20 \times 10^{-3}} = 0.0255$

(c) Amplitude at 1200 rpm

$$\gamma = \frac{1200}{900} = 4/3$$

$$\frac{MX}{me} = \frac{\gamma^2}{\sqrt{(1-\gamma^2)^2 + (2\xi\gamma)^2}}$$

$$\frac{180 \times X}{2 \times 0.092} = \frac{16/9}{\sqrt{\left(1-\frac{16}{9}\right)^2 + \left(2 \times 0.0233 \times \frac{4}{3}\right)^2}}$$

$$X = 2.327 \text{ mm}$$

(d) $$\tan\phi = \frac{2\xi\gamma}{1-\gamma^2}$$

$$\phi = \tan^{-1}\frac{2\xi\gamma}{1-\gamma^2} = -5^\circ \text{ or } 175^\circ$$

Q. A diesel engine of single cylinder has mass 500kg and is mounted on M.S. chassis frame. The static deflection due to weight of chassis is 2.5mm. The reciprocating mass of an engine mounts to 20kg and stroke of the engine is 180mm. A dashpot with a damping coefficient of 2000 N/m/s is also used to dampen the vibration. In the steady state of vibration. Determine -

(i) The amplitude of vibration of driving shaft rotates 400 rpm.

(ii) The speed of the driving shaft when the resonance occurs.

Soln

$M = 500\,kg$ $\quad \delta_{st} = 2.5\,mm$ $\quad m_1 = 20\,kg$ $\quad r = \frac{180}{2} = 90\,mm$

$C = 2000\,N/m/s$, $N = 400\,rpm$.

Stroke = 2r (Reciprocating engine)

(i) $\delta_{st} = 0.0025\,m$.

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 400}{60} = 41.88\ rad/sec.$$

as $K\delta_{st} = mg$

$$K = \frac{mg}{\delta_{st}} = \frac{500 \times 9.81}{0.0025} = 1.962 \times 10^{6}\ N/m.$$

centrifugal force due to reciprocating parts

$$F_0 = m_1 r \omega^2$$

$$= 20 \times (41.88)^2 \times 0.09 = 3157\,N$$

Amplitude

$$X = \frac{F_0}{\sqrt{(C\omega)^2 + (K - m\omega^2)^2}} = \frac{\cancel{3157}}{\cancel{(2000 \times 41.88)^2 - (K - m\omega^2)^2}}$$

$$= \frac{3157}{\sqrt{(2000 \times 41.88)^2 + \{1.962 \times 10^6 - 500\,(41.88)^2\}^2}}$$

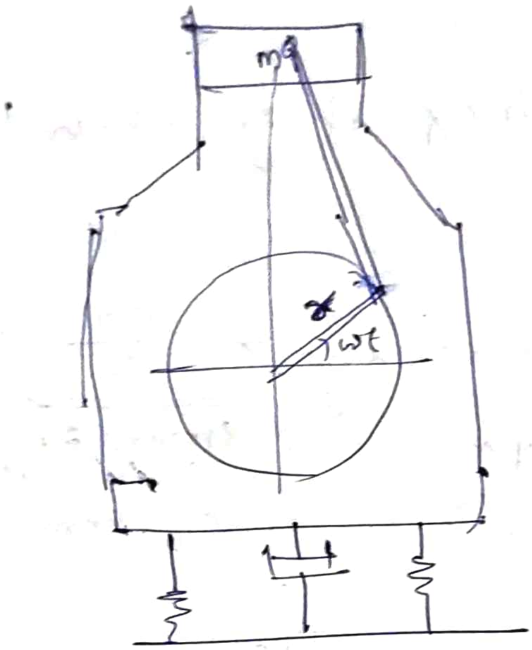
$$= 2.9 \times 10^{-3} = 0.0029\ M$$

(ii) At resonance, $\omega = \omega_n = \sqrt{K/M} = \sqrt{\frac{1.962 \times 10^6}{500}} = 62.64$ rad/sec

$$\omega = \frac{2\pi N}{60}, \quad N = \frac{60\omega}{2\pi}$$

$$= \frac{60 \times 62.64}{2\pi} = 598.16\ rpm.$$

Reciprocating unbalance



$$\underline{F_0 = M_1 r \omega^2}$$

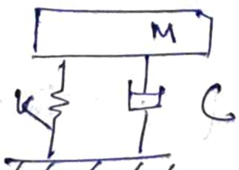
stroke = $L = 2r$

Vibration Isolation

Vibration isolation system attempts either to protect a delicate object from excessive vibration transmitted to it from its supporting structure or to prevent vibratory forces generated by M/C from being transmitted to its surrounding.

When the M/C is directly bolted to foundation or floor, the foundation or floor is subjected to vibrating load and force and transmitted to foundation cause wear and tear of bolts and ultimately the bolts get sheared and also the foundation gets damage.

That is ~~why~~ why an isolation system i.e. an elastic or resilient member (spring and damper) is placed between M/C and rigid foundation. These elastic members are called isolator.

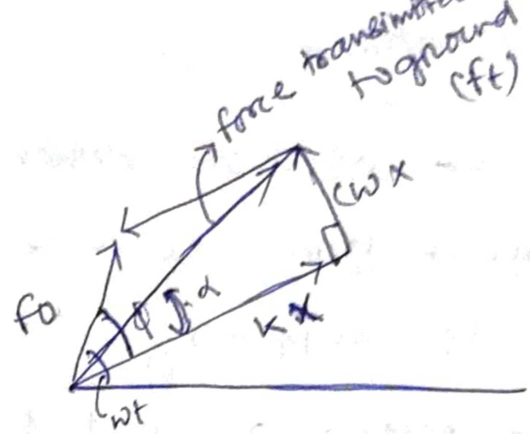


The equation of motion of M/C mass can be written as

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t$$

Maximum amplitude

$$X = \frac{F_0/k \text{ or } (x_{stat})}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \quad \text{or} \quad \frac{F_0/k}{\sqrt{(k-m\omega^2)^2 + (2\xi r)^2}}$$



The force transmitted (f_T) through the spring and damper, as from the vector diagram.

$$f_T = \sqrt{(kx)^2 + (c\omega x)^2} = X\sqrt{k^2 + (c\omega)^2}$$

$$\text{or} = \frac{f_0}{\sqrt{(k - m\omega^2)^2 + (\cancel{2\xi\omega})^2}} \times (k^2 + c\omega^2)$$

$$\left(\text{where } X = \frac{f_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}\right)$$

$$\text{or} \quad \boxed{T_R = \frac{f_T}{f_0} = \sqrt{\frac{1 + (2\xi\,\omega/\omega_n)^2}{\left(1 - (\omega/\omega_n)^2\right)^2 + (2\xi\,\omega/\omega_n)^2}}}$$

where T_R = force transmitted, also called (Transmissibility)

*- The transmissibility T_R is defined as the ratio of the transmitted force to that of disturbing / exciting / driving force.

at resonance $T_R = \dfrac{\sqrt{1 + (2\xi)^2}}{2\xi}$

when damper is not present, $\xi = 0$

$$TR = \frac{1}{\pm\sqrt{1-r^2}}$$

Cases

i) when $r < \sqrt{2}$, TR is more than 1, i.e. transmitted force is greater than exciting force.

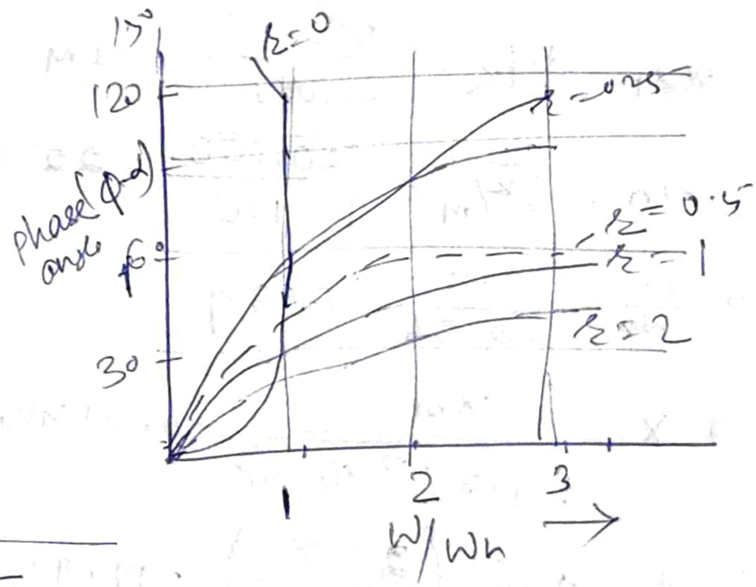
ii) $r > \sqrt{2}$, $TR < 1$, Transmitted force increases as damping force increases.

iii) when $r > 1$, transmitted force is infinite.

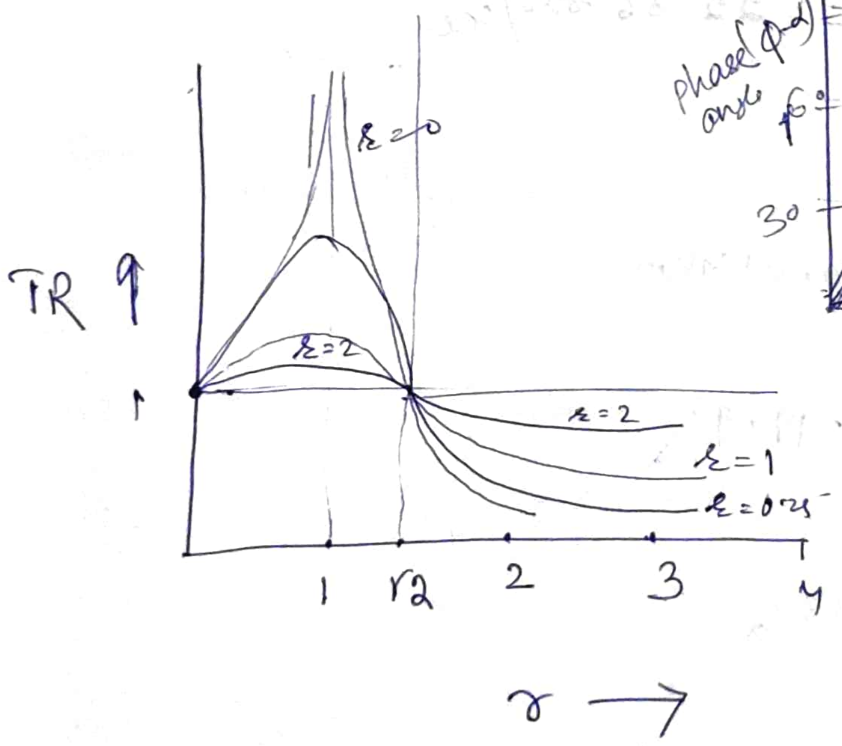
iv) when $r = \sqrt{2}$, TR is unity and is independent of the amount of damping.

v) when $r = 0$, all the curve starts at $TR = 1$, for different value of ξ.

Phase-lag

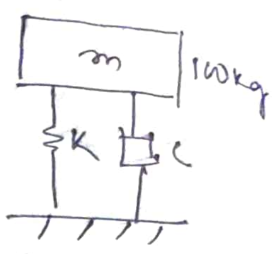


Response curve



Q

A system shown in fig is subjected to harmonic force $F = 500 \sin 13.2t$. The value of spring stiffness is $50{,}000$ N/m and damping factor $\xi = 0.2$. For steady state vibration of the system determine. a) Amplitude of motion of the system



(b) phase angle (c) Transmissibility.

(d) Maximum dynamic force transmitted to the foundation.

(e) Maximum velocity of motion.

Sol^n

~~F = 500 sin wt~~

$F = 500 \sin 13.2t$, equating with, $F_0 \sin \omega t$

$F_0 = 500$, $\omega = 13.2$, $K = 50{,}000$ N/m, $\xi = 0.2$

$m = 100$ kg

$$x_{st} = F_0/K = \frac{500}{50{,}000} = 0.01 \text{ m}$$

$$\omega_n = \sqrt{K/m} = \sqrt{\frac{50{,}000}{100}} = 22.36 \text{ rad/sec.}$$

$$r = \frac{\omega}{\omega_n} = \frac{13.2}{22.36} = 0.59$$

a) $$X = \frac{x_{st}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} = 0.0144 \text{ m.}$$

(b) $$\phi = \tan^{-1}\left(\frac{2\xi r}{1-r^2}\right) = 19.9^\circ$$

c) $$TR = \frac{\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \; 1.482$$

(d) $F_{TR} = F_0 \times TR = 1.482 \times 500 = 741$ N

(e) Max^m velocity $= \dot{x}_{max} = \omega X = 0.19$ m/s

Q3 A M/c of mass 1250 kg is mounted on spring of stiffness 2 MN/m and a dashpot giving damping factor of 0.15. Piston of mass 30 kg moves up and down in the M/c with a speed 540 cpms and a stroke of 0.45. The motion be considered to be harmonic (a) find out the amplitude of motion of the M/c. (b) phase angle of the motion (c) the amplitude of force transmitted to foundation (d) phase angle of transmitted force

Soln $M = 1250\,kg$, $k = 2\,MN/m = 2\times10^6\,N/m$, $\xi = 0.15$

$m_1 = 30\,kg$ ~~N = 50 CPM~~ $= n = 540$ cycles/sec.

$w = \frac{2\pi n}{60} = 56.55$

$e = r = L/2 = \frac{0.45}{2}$

(a) $w_n = \sqrt{K/m} = \sqrt{\frac{2\times10^6}{30}} = 40$ rad/sec.

$\frac{m_1}{M} = \frac{30}{1250} = 0.024$, $r = \frac{w}{w_n} = \frac{56.55}{40} = 1.41375$

$$X = \frac{\frac{m_1}{M} e r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} = 9.95$$

(b) $\phi = \tan^{-1}\left(\frac{2\xi r}{1-r^2}\right) = -23.01$

(c) $$\frac{F_{TR}}{F_0} = \frac{\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} = 21610\,N$$

(d) $\alpha = \tan^{-1}(2\xi r) = 22.983^\circ$ (let α = angle of F_R to spring force)

$\phi - \alpha = -23.01 - 22.983 = -45.993$

Q. A M/C of 100 kg mass is suspended on spring of total stiffness 700 KN/m and has unbalanced rotating element, which results a disturbing force of 350 N at a speed of 3000 rev/min, Assuming the damping factor of $\xi = 0.20$. Determine its

(a) Amplitude of motion due to unbalance

(b) Transmissibility (c) Transmitted force

Sol^n^ $m = 100\,kg$, $K = 700\,KN/m$ $f_0 = 350\,N$, $N = 3000\,rpm$

$\xi = 0.20$

As $K\delta_{st} = mg$, $\delta_{st} = \frac{mg}{K} = \frac{100 \times 9.81}{700 \times 10^3} = 1.401\,mm$

Natural frequency $f_n = \frac{1}{2\pi}\sqrt{g/\delta_{st}} = 13.32\,Hz$.

$$f = \frac{\omega}{2\pi} = \frac{2\pi N}{60} = \frac{2\pi \times 3000}{60} = 50\,Hz$$

(a)
$$X = \frac{f_0/k}{\sqrt{\left(1 - \left(\frac{50}{13.32}\right)^2\right)^2 + \left(2 \times 0.20 \times \frac{50}{13.32}\right)^2}}$$

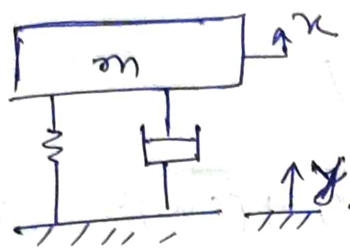
$$= 3.79 \times 10^{-5}\,m = 0.0379\,mm$$

(b) Transmissibility $= TR = \frac{\sqrt{1 + (2\xi\gamma)^2}}{\sqrt{(1-\gamma^2)^2 + (2\xi\gamma)^2}} = 0.137.$

(3) $F_{TR} = \text{disturbing} \times T_r$

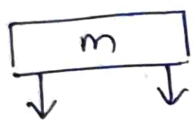
$$= 350 \times 0.137 = 47.89\,N$$

excited by motion of the support. i.e the base excitation (Earth quake), vehicle rolls on wavy road, engine mountained on a vibrating plat-form.



Let y be the harmonic displacement of the support point and measures the displacement x of mass 'm' from the inertial reference.

$y = Y \sin \omega t$ (cause of excitation)



$\downarrow \ddot{x}$ = (absolute accl^n)

$\underline{c(\dot{x}-\dot{y})}$ (Damping factor Relative to each other) $\quad \underline{K(x-y)}$ Relative displacement

The differential equation of motion

$$M\ddot{x} + c(\dot{x}-\dot{y}) + K(x-y) = 0 \quad \text{--- (1)}$$

Let $x - y = z(t)$, relative displacement between excitation mass.

$$x = y + z$$

Then, the equation of motion changes to

$$M(\ddot{y}+\ddot{z}) + c\dot{z} + Kz = 0$$

where $y = Y \sin \omega t$ has been assumed for motion of base

$$\ddot{y} = -Y\omega^2 \sin \omega t$$

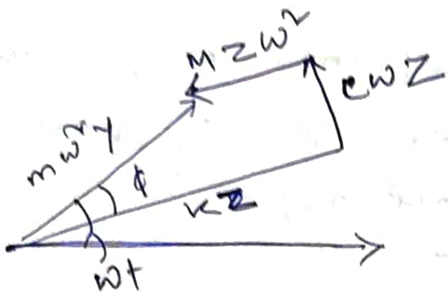
Hence the motion equation changes to

$$M\ddot{z} + c\dot{z} + kz = m\omega^2 y = m\underbrace{\omega^2 Y}_{\text{equivalent to } me\omega^2} \sin \omega t$$

$= \cancel{mep}^2$

Thus the solution can be written as

$$z = Z \sin(\omega t - \phi)$$



$$\boxed{Z = \frac{m\omega^2 Y}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}}$$

$$\tan\phi = \frac{c\omega}{k - m\omega^2} = \frac{2\xi(\omega/\omega_n)}{1 - (\omega/\omega_n)^2} = \frac{2\xi r}{1 - r^2}$$

Again the eqn (2)

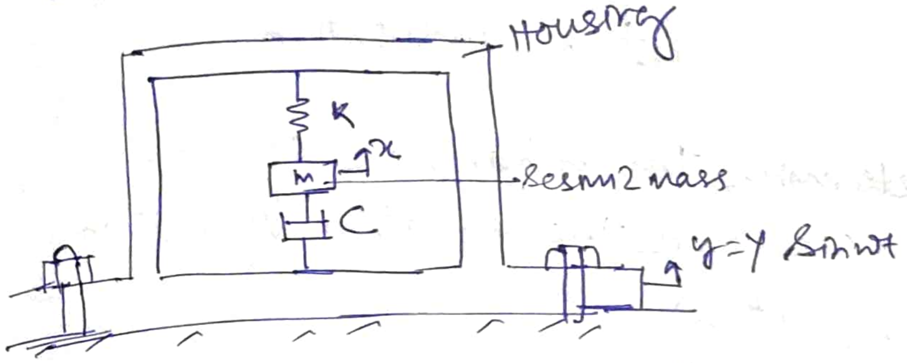
$$Z = \frac{r^2 Y}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

$$\boxed{\frac{Z}{Y} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}}$$

where $r = \omega/\omega_n$

vibration measuring instrument

seismic Instrument - (vibrometer and acceleration)



The mass remain stationary, while supporting case moves with the vibrating body. Such instrument is called Seismometer.

The seismic instrument is a device which has functional form of mass connected through spring and damper arrangement to be housing frame. The frame is connected to the source of vibration where the characteristic to be measured. The mass tends to remains fixed in its spatial position. So the vibrational motion is registered as relative displacement between the mass and housing frame. Depending on the frequency range utilised, displacement, velocity or acceleration is indicated by relative motion of the suspended mass w.r.t the case.

The seismic instrument may be used for either displacement or acceleration measurements by proper selection of mass, spring and damper combination. In general a large mass and soft spring are used for vibrational displacement and while relatively small mass and stiff spring are used for acceleration indication.

To determine the behaviour of this instrument, we consider the equation of motion (Suppose motion equation)

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

or $$m\ddot{z} + c\dot{z} + kz = m\omega^2 y \sin\omega t$$

and the solution of this equation is

$$z = Z \sin(\omega t - \phi)$$

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

vibrometer - vibrometer is an instrument used to measure the displacement of vibrating body

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

when r is very high $(r > 1)$, $\xi = 0$

Then the above equation is ~~reduce~~ written as

$$\frac{Z}{Y} = \frac{r^2}{\pm(1-r^2)} \simeq 1$$

in this case $Z \simeq Y$, so the relative amplitude equal to the amplitude of vibrating body.

→ In this instrument damping is kept as small as possible

→ It is known as low frequency instrument.

→ The average value of ω_n for vibrometers is 4 Hz.

→ Used to record building vibrations or huge structure like railway bridge.

Accelerometer – Accelerometer is used to measure the acceleration of the vibrating body.

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

where r is very small & $\xi = 0$

$$\frac{Z}{Y} = r^2 \quad \text{or} \quad \boxed{Z = Y r^2}$$

$$\text{or} \quad Z = Y \cdot \left(\frac{\omega}{\omega_n}\right)^2 = \omega^2 Y \times \frac{1}{(\omega_n)^2}$$

Here $\frac{1}{(\omega_n)^2}$ is a constant of the instrument, and independent of excitation frequency 'ω', $\omega^2 Y$ is the acceleration, so measuring 'Z' we can get acceleration directly.

→ It is used in high frequency transducer.

Q. A vibrometer indicates 1% error in measurement and its natural frequency is 4 Hz. If the lowest frequency that can be measured is 36 Hz. find the value of damping factor.

Soln Since the reading recorded by the vibrometer is Z, so as per data given.

$$Z = 1.01Y$$

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$= 0.01 = \frac{(36/4)^2}{\sqrt{\left(1-\left(\frac{36}{4}\right)^2\right)^2 + \left(2\xi\,\frac{36}{4}\right)^2}}$$

$$\boxed{\xi = 0.313}$$

Q. A vibration measuring instrument is used to find the displacement, velocity and acceleration of a m/c running at 120 rpm. If the natural frequency of the instrument is 5 Hz and it shows 0.004 cm.. what are the three reading? Assume no damping.

Soln $\omega_n = 5\,Hz = 5 \times 2\pi = 10\pi$ rad/sec.

$$\frac{Z}{Y} = \frac{r^2}{1-r^2}, \quad r = \frac{\omega}{\omega_n} = \frac{2\pi \times 120}{60 \times 10\pi} = 0.40$$

$$Z = 0.004 \text{ cm.}$$

Displacement, $Z = Y\left(\frac{1-r^2}{r^2}\right) =$

$= 0.004 \times \frac{(1-(0.4)^2)}{(0.4)^2} = 0.021 cm.$

velocity $= \omega Y = \left(\frac{2\pi N}{60}\right) Y = \left(\frac{2\pi \times 120}{60}\right) \times 0.021$

$= 0.26$ cm/sec

Acceleration $- \omega^2 Y = \omega \times (\omega Y) = \left(2\pi \times \frac{120}{60}\right) \times 0.26 = 3.265$ cm/s^2

Q An accelerometer is made by mounting a 2.5kg block on rubber isolator that have combined spring scale of 35100 N/m and a viscous damping factor ratio $\xi = 0.7$. The amplitude read on the dial indicator of vibrations occurring at 200 CPM is 0.1 mm. What is the maximum acceleration of the member to which the accelerometer is attached.

Sol $m = 2.5$ kg $\quad K = 35100$ N/M $\quad \xi = 0.7 \quad Z = 0.1$mm

$N = 200$ cycles/min.

$\omega = \frac{2\pi(200)}{60} = 20.94$ rad/sec.

$\omega_n = \sqrt{K/m} = \sqrt{\frac{35100}{2.5}} = 118.5$ rad/sec.

$r = \omega/\omega_n = \frac{20.94}{118.5} = 0.177$

$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$, $\left(\frac{0.1}{Y}\right)^2 = \frac{(0.177)^2}{\sqrt{(1-(0.177)^2)^2 + (2\times 0.7\times 0.177)^2}}$

$Y = 3.19$ mm.

Maximum acceleration $(\ddot{x})_{max} = \omega^2 Y$

$= (20.94)^2 \times 3.19$

$= 1.399$ m/s.

Sharpness resonance

The magnification factor at resonance

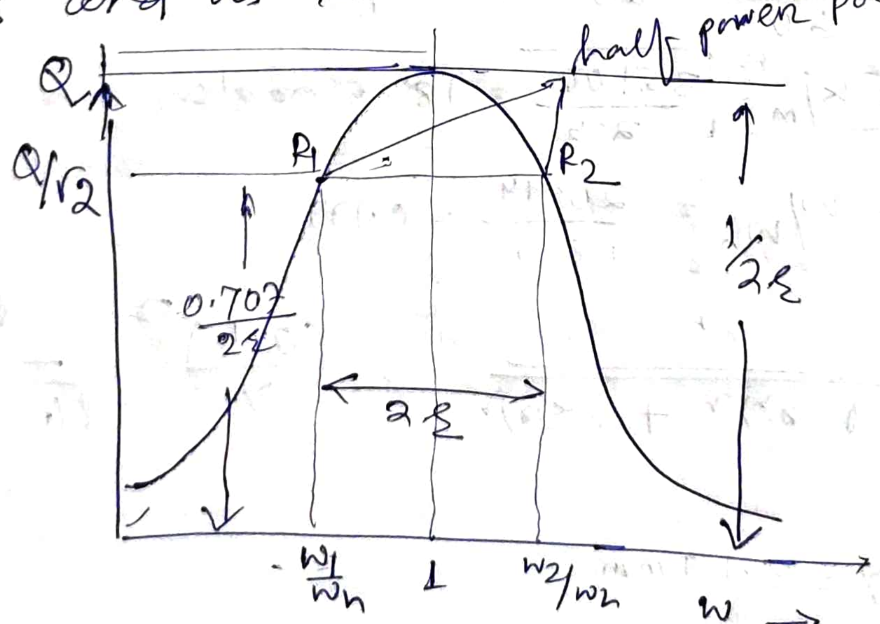
$$M_f = \frac{X_{Res}}{X_{st}} = \frac{1}{2\xi}$$

& $X_{Res} = \frac{F_0/K}{2\xi}$ —— ①

The value of magnification factor at resonance called as quality factor (Q)

The quality factor $\boxed{Q = \frac{1}{2\xi}}$

Now we seek two frequency on either side of resonance (reffered to as side bands), where $X = \frac{X_{Res}}{\sqrt{2}} \approx 0.707 X_{Res}$, The pt. where Amplitude ratio these points referred to the half power points and as shown.



* The points where the amplitude ratio is $\frac{R}{\sqrt{2}}$ =0.7 are known as half power points.

$$X = \frac{X_{res}}{\sqrt{2}} = \frac{f_0/k}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$= \frac{\cancel{f_0/k}}{2\xi} \cdot \frac{1}{\sqrt{2}} = \frac{\cancel{f_0/k}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

squaring both side

$$\frac{1}{2} \cdot \frac{1}{(2\xi)^2} = \frac{1}{(1-r^2) + (2\xi r)^2}$$

expanding

$$r^4 - 2(1-2\xi^2) r^2 + (1 - 8\xi^2) = 0$$

or $\quad r^2 = (1-2\xi^2) \pm 2\xi\sqrt{1-\xi^2} \quad$ — (2)

Assuming $\xi << 1$ & neglecting higher order term of ξ,

$$r^2 = 1 \pm 2\xi$$

or $\boxed{\left(\frac{\omega}{\omega_n}\right)^2 = 1 \pm 2\xi}$ — (3) or we can write

$\boxed{\left(\frac{\omega_2}{\omega_n}\right)^2 = 1 + 2\xi}$ — (4), & $\boxed{\frac{\omega_1}{\omega_n} = 1 - 2\xi}$ — (5)

Subtracting eqn (4) − (5)

$$4\xi = \frac{\omega_2^2 - \omega_1^2}{\omega_n^2} \quad \text{— (6)} \quad \text{Since } \frac{\omega_1 + \omega_2}{2} = \omega_n$$

from eqn (6)

$$\overset{2}{\cancel{4}}\xi = \frac{\cancel{2}(\omega_2 - \omega_1)}{\omega_n}$$

$\boxed{\text{ie } 2\xi = \frac{\omega_2 - \omega_1}{\omega_n}}$ — Bandwidth — Difference of frequency ratio at power point called band width

* The quality Q related to damping that is a measure of the sharpness resonance is defined as

$$Q = \frac{1}{\text{Band width}} = \frac{\omega_n}{\omega_2 - \omega_1} = \frac{f_n}{f_2 - f_1} = \frac{1}{2\xi}$$

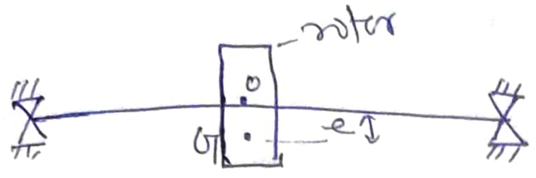
Whirling of shaft

When the gear or pulleys are put on the shaft, the CG of pulley or gear doesnot coincide with the centre line of the bearings or with the axis of shaft, when the shaft is stationary. If the CG of the pulley or gear doesnot coincide with centre line of the when in rotation, the shaft is subjected to centrifugal force. This force will bend the shaft which will further increase the distance of CG of the pulley or gear from the axis of rotation. when the centrifugal force further increases ultimately the shaft fails. The bending of shaft not only depends upon the value of eccentricity but also depends upon the speed at which the shaft rotates.

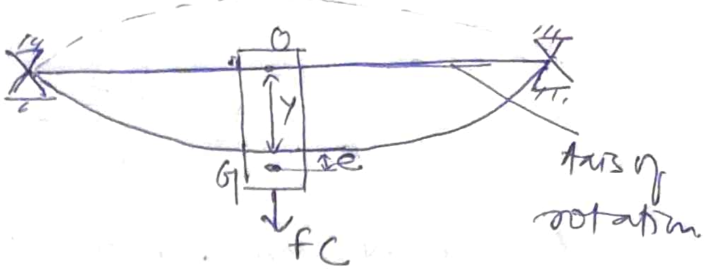
Critical Speed | whipping speed | whirling speed :-

The critical speed of the rotating shaft is speed at which the shaft starts to vibrate ~~vibrati~~ violently in the trans-verse direction. Also, It is termed as the speed at which resonance occurs.

Consider a shaft of negligible mass carrying a rotor. The point O' on the shaft axis and G is the CG of the rotor. when the shaft is stationary, the centreline of the bearing and the axis of the shaft coincides.



(a) when the shaft is stationary



(b)

(when in rotation)

The fig (b) shows when the shaft rotating at uniform speed 'ω' rad/sec.

Let m = Mass of the rotor

e = Initial distance of CG of rotor from the centre line of the shaft axis, while shaft is stationary

y = Additional deflection of CG of rotor when the shaft starts rotating at ω rad/sec.

K = stiffness of the shaft.

Considering the equilibrium of forces acting on the disc and shaft.

The centrifugal force ~~du~~ = Restoring force due to Lateral stiffness of shaft.

$$F_C = m\omega^2(y+e) = Ky$$

$$m\omega^2 y + m\omega^2 e = ky$$

$$y(k - m\omega^2) = me\omega^2$$

$$y = \frac{me\omega^2}{k - m\omega^2} = \frac{\omega^2 e}{k/m - \omega^2} = \frac{\omega^2 e}{\omega_n^2 - \omega^2} = \frac{\left(\frac{\omega}{\omega_n}\right)^2 e}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

- A little consideration will show that when $\omega > \omega_n$, the value of y will be -ve and the shaft deflects in opposite direction as shown in dotted line.

In order to have the value of y always positive, both, both +ve & (-ve) value are taken

$$y = \pm \frac{\omega^2 e}{\omega_n^2 - \omega^2}$$

* for $\omega = \omega_n$, y becomes ∞, this speed is Called critical or whirling speed $\underline{\omega_c}$

∴ critical / whirling speed

$$\omega_c = \omega_n = \sqrt{k/M} = \sqrt{\frac{g}{\delta_{st}}} \text{ rad/sec}$$

If N_c is the critical or whirling speed in rps.

$$\omega_c = 2\pi N_c$$

$$N_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi}\sqrt{k/M} = \frac{1}{2\pi}\sqrt{g/\delta_{st}} \text{ rps.}$$

Thus the critical sped is the same as natural ~~speed~~ frequency of transvibration.

critical speed with damping

$$\frac{y}{e} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$\phi = \tan^{-1}\frac{2\xi r}{1-r^2}$$

Q A disc having mass of 6kg is mounted half way as 2cm dia shaft supported at the ends by two bearings. The length of the bearing support is 50cm. Due to manufacturing defect, the CG of the disc is 0.01 away from the geometric centre of the disc. If the system rotates at 2000 rpm. find the amplitude of steady state and dynamic force transmitted to the bearing. $E = 1.96 \times 10^{11}\ N/m^2$,

Given $m = 6kg$ $d = 2cm$ $l = 50cm$ $e = 0.01mm$, $N = 2000$ rpm

Assuming it to be simply supported

$$\delta = \frac{Wl^3}{48EI} = \frac{mgl^3}{48EI}, \text{ as } \boxed{\frac{Mg}{\delta}} = \boxed{\frac{48EI}{l^3}} = K$$

$$K = \frac{48 \times 1.91 \times 10^{11} \times \frac{\pi}{16}(0.02)^4}{(0.5)^3} = 59112\ N/m$$

$$w_n = \sqrt{K/M} = \sqrt{\frac{59112}{6}} = 313.8\ rad/sec.$$

$$w = \frac{2\pi N}{60} = \frac{2\pi \times 2000}{60} = 209.4\ rad/sec.$$

$$\text{Amplitude } y = \frac{w^2 e}{w_n^2 - w^2} = 8.02 \times 10^{-6}\ m$$

$$\text{Dynamic load on each bearing} = \frac{ky}{2}$$

$$= \frac{59112 \times 8.02 \times 10^{-6}}{2} N = 2.37 N$$

Q. A rotor has a mass of 12kg and is mounted midway on a 24mm diameter horizontal shaft supported at the ends by two bearings. The bearings are one meter apart. The shaft rotates at 2400 rpm. If the centre of the mass of the rotor is 0.11 met away from the geometric centre of the rotor due to a certain manufacturing error. Find the amplitude of the steady state vibration and dynamic force transmitted to each bearing. Take $E = 200 \text{ GN/m}^2$

Sol^n $m = 12\,kg$ $\quad d = 24\,mm$ $\quad l = 1\,m$ $\quad N = 2400\,rpm$

$e = 0.11\,m$ $\quad E = 200\,kN/m^2$

$$I = \frac{\pi}{64} d^4 = 16.3 \times 10^{-9}\,m^4$$

$$\delta = \frac{mgl^3}{48EI} = 0.000752\,m$$

$$\omega_n = \sqrt{g/\delta} = 114.2 \text{ rad/sec.}$$

$$\omega = \frac{2\pi N}{60} = 251 \text{ rad/sec.}$$

Amplitude $y = \dfrac{e}{\left(\frac{\omega_n}{\omega}\right)^2 - 1} = -0.000139$

The $-ve$ sign indicates that the displacement is out of phase with centrifugal force.

Dynamic force on the bearing $ky = m\omega_n^2 y = 21.7$

Total load on each bearing

$$= \frac{mg}{2} + \frac{ky}{2} = 69.7\,N$$

Two Degrees of freedom System (2DOF)

The System which requires two co-ordinate independently to describe its motion completely, it is called two degree of freedom System.

The general rule for the computation of the number of degree of freedom can be stated as–

No. of DOF = no. of masses in the System × no. of Possible type of motion of each mass.

Normal mode or principal mode - During free vibration at one of the natural frequencies, the amplitude of 2 (dof) (co-ordinate) are the related in a specific manner and the configuration is called normal mode / Principal mode / natural mode.

→ Here the Amplitude ratio is to be unity.

→ A 2 DOF System has two normal modes of vibration corresponding to two natural frequency.

principal co-ordinate - when each equation of motion contains only one unknown quantity, then the equation of motion can be solved indipendently of each other, such set co-ordinate is called principal co-ordinate.

coupled - when both the co-ordinate appear in each equation of motion then the 2 DOF system is said to be coupled.

Two types of coupling - i) stiffness or static - due to static displacement

ii) mass or dynamic coupling - due to inertia force

* stiffness matrix exits when stiffness matrix is non-diagonal.
* Dynamic coupling exists, when mass matrix is nondiagonal.

Proportional

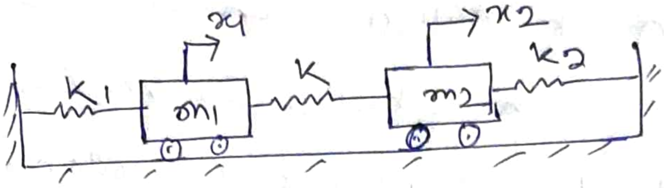
If the damping matrix is diagonal then the damping is said to be proportional.

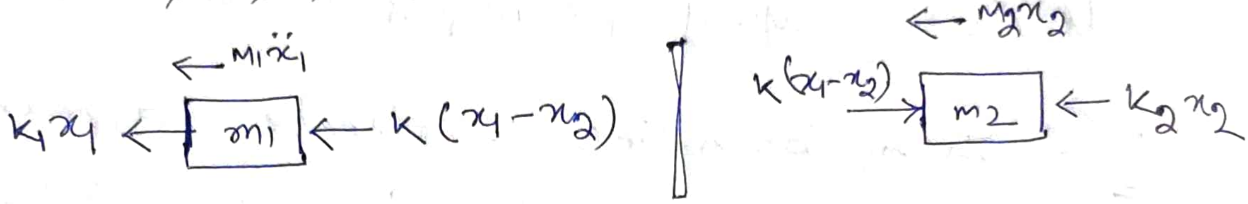
Semidefinite system

when one of the frequencies of 2 DOF system is equal to zero, then it is said to be semidefinite system.

Normal mode vibration :

(Undamped free vibration of 2DOF)





The differential equation of motion

$$m_1\ddot{x}_1 + k(x_1 - x_2) + k_1 x_1 = 0 \quad \text{---(1)}$$

$$m_2\ddot{x}_2 - k(x_1 - x_2) + k_2 x_2 = 0 \quad \text{---(2)}$$

Rearranging the above equation

$$m_1\ddot{x}_1 + (k + k_1)x_1 - k x_2 = 0 \quad \text{---(3)}$$

$$m_2\ddot{x}_2 + (k + k_2)x_2 - k x_1 = 0 \quad \text{---(4)}$$

For periodic motion Assume

$$x_1 = A_1 \sin(\omega t + \phi) \quad \text{---(5)}$$

$$x_2 = A_2 \sin(\omega t + \phi) \quad \text{---(6)}$$

Substituting the value of $x_1, x_2, \ddot{x}_1$ & $\ddot{x}_2$ in eqn (3)

$$-m_1 A \omega^2 \sin(\omega t + \phi) + (k_1 + k) A_1 \sin(\omega t + \phi) - k A_2 \sin(\omega t + \phi) = 0$$

Rearranging

$$A_1(k + k_1 - m_1\omega^2) - A_2 k = 0 \quad \text{---(7)}$$

Similarly equation (4) becomes

$$-A_1 k + A_2(k + k_2 - m_2\omega^2) = 0 \quad \text{---(8)}$$

Above equations are homogeneous linear algebraic equation for non-trivial solution, we get -

$$\begin{bmatrix} k+k_1-m_1\omega^2 & -k \\ -k & k+k_2-m_2\omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} k+k_1-m_1\omega^2 & -k \\ -k & k+k_2-m_2\omega^2 \end{vmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Expanding and solving it, we get

$$\omega^4 - \left[\frac{k+k_2}{m_2} + \frac{k_1+k_2}{m_1}\right]\omega^2 + \frac{k_1k_2+kk_1+kk_2}{m_1m_2} = 0 \quad \text{——— (9)}$$

This frequency equation is a quadratic ~~equation~~ in ω^2 we get two values of ω^2.

<u>Special case</u> :-

If $k_1 = k_2 = k$, $m_1 = m_2 = m$, then the above equation can be written as

$$\omega^4 - \left[\frac{2k}{m} + \frac{2k}{m}\right]\omega^2 + \frac{3k^2}{m^2} = 0$$

$$\omega^4 - \frac{4k}{m}\omega^2 + \frac{3k^2}{m^2} = 0$$

$$\omega^2 = \frac{\frac{4k}{m} \pm \sqrt{\frac{16k^2}{m^2} - \frac{12k^2}{m^2}}}{2}$$

or $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{\frac{3k}{m}}$

Where ω_1 & ω_2 are the frequencies of 1st and 2nd mode only.

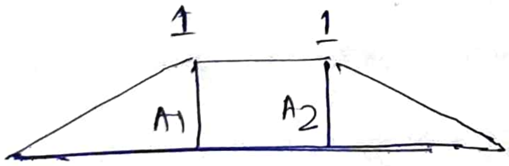
Amplitude ratio can be written from equation (7) & (8)

$$\left.\frac{A_1}{A_2}\right]_{w=w_1} = \frac{k}{2k-k} = 1 \quad \left[\text{Putting } w_1 = \sqrt{k/m},\ k_1 = k_2 = k\right]$$

$$\left.\frac{A_1}{A_2}\right]_{w=w_2} = \frac{2k-3k}{k} = -1 \quad \left[\text{Putting } w_2 = \sqrt{\frac{3k}{m}},\ k_1 = k_2 = k\right]$$

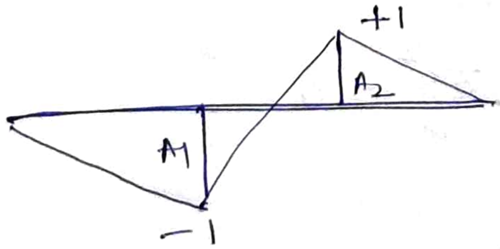
From the above amplitude ratios that for the 1st two mode, the two masses moves in same phase with equal amplitude and for the 2nd mode two masses moves in (opposite) or out of phase with equal amplitude

Mode shapes If one amplitude chosen one or any number the amplitude ratio is normalise to that number.



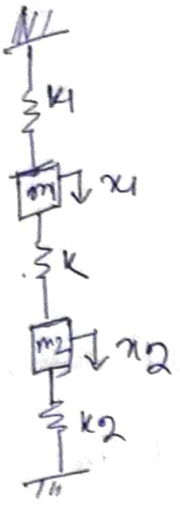
If $(x_1)_1 = +1$
$(x_2)_1 = +1$

$w_1 = \sqrt{k/m}$ (mode shap)



If $(x_1)_2 = +1$
$(x_2)_2 = -1$, then mode shape.

Principal co-ordinate



The equation of motion of the above System are

$$M\ddot{x}_1 + kx_1 + k(x_1 - x_2) = 0$$
$$M\ddot{x}_2 + kx_2 - k(x_1 - x_2) = 0$$

on re-arranging

$$M\ddot{x}_1 + 2kx_1 - kx_2 = 0$$
$$M\ddot{x}_2 + 2kx_2 - kx_1 = 0$$

Assumed $x_1 = A_1 \sin \omega_n t$, $m_1 = m_2 = m$
$x_2 = A_2 \sin \omega_n t$,

then $$\begin{bmatrix} 2k - m\omega_n^2 & -k \\ -k & 2k - m\omega_n^2 \end{bmatrix} = 0$$

$$(2k - m\omega_n^2)^2 - k^2 = 0$$

$$2k - m\omega_n^2 = \pm k$$

$$\omega_{n_1} = \sqrt{k/M} \quad \& \quad \omega_{n2} = \sqrt{3k/M}$$

Defining two independent solution as principal co-ordinate and expressed is given by

$$\left.\begin{aligned} x_1(t) &= A_1 \sin\left(\sqrt{k/m}\,t + \alpha_1\right) + A_2 \sin\left(\sqrt{\tfrac{3k}{m}}\,t + \alpha_2\right) \\ x_2(t) &= A_1 \sin\left(\sqrt{k/m}\,t + \alpha_1\right) - A_2 \sin\left(\sqrt{\tfrac{3k}{m}} + \alpha_2\right) \end{aligned}\right\} - (A)$$

Let $Y_1 = A_1 \sin\left(\sqrt{k/m}\,t + \alpha_1\right)$

$Y_2 = A_2 \sin\left(\sqrt{\tfrac{3k}{m}} + \alpha_2\right)$

Since y_1 & y_2 are the harmonic functions, their corresponding equations be

$$\left.\begin{aligned} \ddot{y}_1 + \frac{k}{m} y_1 = 0 \\ \ddot{y}_2 + \frac{3k}{m} y_2 = 0 \end{aligned}\right] \text{——— (B)}$$

These equations represents a 2 DOF system whose natural frequency are $\omega_1 = \sqrt{k/m}$ & $\omega_2 = \sqrt{3k/m}$. Because there is neither static nor dynamic coupling in equation (B), so y_1 & y_2 are called principal co-ordinate.

Substituting the values of ~~equation~~ y_1 & y_2 in equation (A)

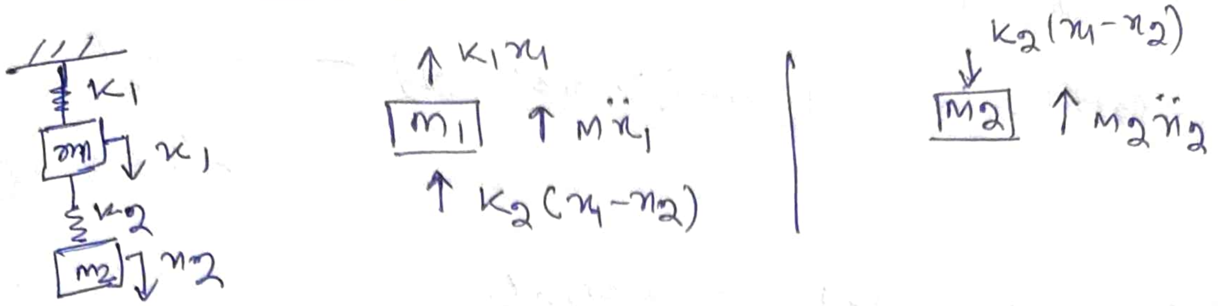
$$x_1 = y_1 + y_2, \qquad x_2 = y_1 - y_2$$

Solving these two equations

$$y_1 = \frac{x_1 + x_2}{2}, \quad y_2 = \frac{x_1 - x_2}{2}$$

where y_1 & y_2 are called principal co-ordinate.

Determine the principal co-ordinate of the system given below.



The equation of motions are

$$M_1\ddot{x}_1 + k_1x_1 + k_2(x_1-x_2) = 0$$

$$m_2\ddot{x}_2 - k_2(x_1-x_2) = 0$$

Let $x_1 = A_1 \sin w_n t$ $\quad x_2 = A_2 \sin w_n t$, then

$$\begin{bmatrix} k_1+k_2-m_1w_n^2 & k_2 \\ -k_2 & k_2-m_2w_n^2 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For $k = k_1 = k_2$, $m_1 = m_2 = m$

$$(2k - mw_n^2)(k - mw_n^2) - k^2 = 0$$

$$w_n^4 - 3k/m \, w_n^2 + k^2/m^2 = 0$$

$$w_n^2 = \frac{1}{2}\left[3k/m \pm \sqrt{\frac{9k^2}{m^2} - \frac{4k^2}{m^2}}\right] = \left[\frac{3 \pm \sqrt{5}}{2}\right] k/m$$

or $w_{n_1} = 0.618\sqrt{k/m}$ $\qquad w_{n_2} = 1.618\sqrt{k/m}$

$$\frac{A_1}{A_2} = 0.618 \Big] \text{ for } w_1$$

$$\frac{A_1}{A_2} = 1.618 \Big] \text{ for } w_{n_2}$$

The general solution is

$$x_1(t) = A_1 \sin(0.618t + \alpha_1) + A_2 \sin(1.618t + \alpha_2)$$

$$x_2(t) = 1.618 A_1 \sin(0.618t + \alpha_1) - 0.618 A_2 (\sin(1.618 + \alpha_2)$$

Defining new set co-ordinate

$$y_1 = A_1 \sin(0.618t + \alpha_1)$$

$$y_2 = A_2 \sin(1.618 + \alpha_2)$$

Then $\ddot{y}_1 + 0.382 y_1 = 0$ as $(0.618^2 = 0.382)$

$\ddot{y}_2 + 2.618 y_2 = 0$ $((1.618)^2 = 2.618)$

Now $x_1 = y_1 + y_2$, $x_2 = 1.618 y_1 - 0.618 y_2$

on solving the above two equations

$$y_1 = 0.2764 x_1 + 0.4472 x_2$$

$$y_2 = 0.7236 x_1 - 0.4472 x_2$$

where y_1 & y_2 are the principal co-ordinate.

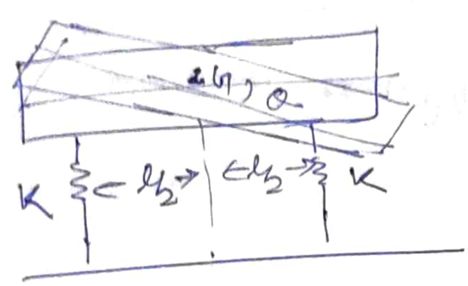
$\left.\frac{A_1}{A_2}\right]_{\omega_1} = r_1$ $\left.\frac{A_2}{A_2}\right]_{\omega_2} = r_2$

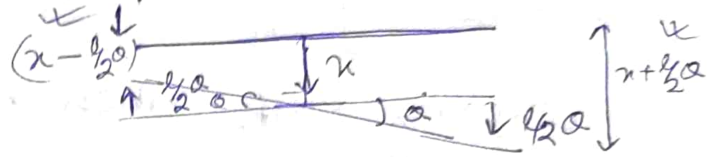
$$x_1 = A_1 \sin(\omega_1 t + \alpha_1) + A_2 \sin(\omega_2 t + \alpha_2)$$

$$x_2 = \frac{1}{r_1} A_1 \sin(\omega_1 t + \alpha_1) + \frac{1}{r_2} A_2 \sin(\omega_2 t + \alpha_2)$$

Principal co-ordinate :-

When each equation of motion contains only one unknown quantity, then the equation of motion can be solved independently each other. Such set of co-ordinate is called principal co-ordinate.





$$m\ddot{x} + k(x - \frac{l\theta}{2}) + k(x + \frac{l}{2}\theta) = 0$$

$$m\ddot{x} + 2kx = 0 \quad \text{———} \; (1)$$

$$\omega n_1 = \sqrt{2k/m}$$

Also the inertia torque.

$$I\ddot{\theta} + k(x - \frac{l\theta}{2}) \cdot \frac{l}{2} + k(x + \frac{l\theta}{2})\frac{l}{2} = 0$$

$$I\ddot{\theta} + \frac{kl^2}{2}\theta = 0$$

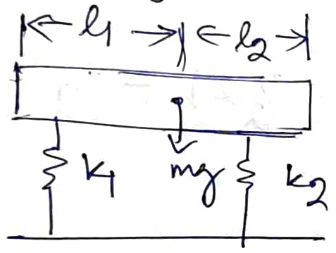
Hence natural frequency

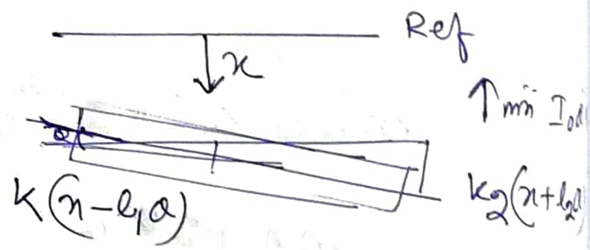
$$\omega n_2 = \sqrt{\frac{kl^2}{2I}}$$

Co-ordinate coupling

Coupling :- when both the co-ordinate appear in each equation of motion then the 2 DOF system said to be coupling.

Static coupling :- (stiffness coupling) - static coupling exists when stiffness matrix is non-diagonal.





Here co-ordinates are x & θ.

The equation of motions

$$m\ddot{x} + k_1(x_1 - l_1\theta) + k_2(x + l_2\theta) = 0$$

The moment equation about CG

$$I_0\ddot{\theta} - k_2(x + l_2\theta)l_2 + k_1(x - l_1\theta)l_1 = 0$$

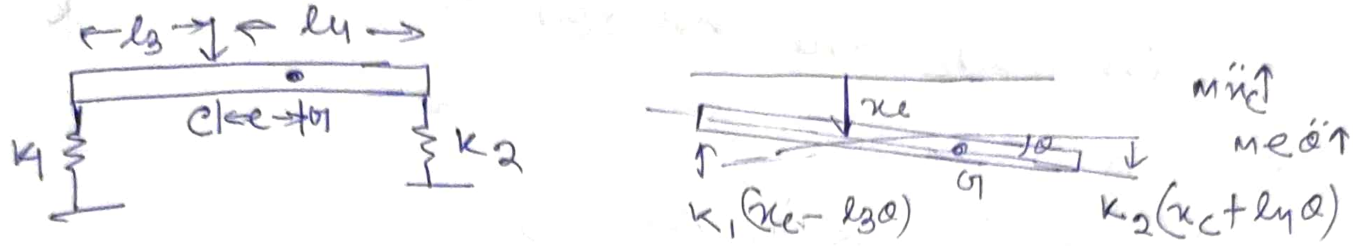
Re-arranging above two equations

$$\begin{bmatrix} m & 0 \\ 0 & I_0 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & k_2l_2 - k_1l_1 \\ k_2l_2 - k_1l_1 & k_1l_1^2 + k_2l_2^2 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$$

If $k_1l_1 = k_2l_2$, the coupling disappears and we obtained uncoupled x and θ vibration.

Dynamic coupling

There is some point c along the bar where a force applied normal to the bar produces pure translation.



$$m\ddot{x}_c + k_1(x_c - l_3\theta) + k_2(x_c - l_4\theta) + me\ddot{\theta} = 0$$

& moment about 'c'

$$I_0\ddot{\theta} + k_2(x_c - l_4\theta)l_4 - k_1(x_c - l_3\theta)l_3 + me\ddot{x}_c = 0$$

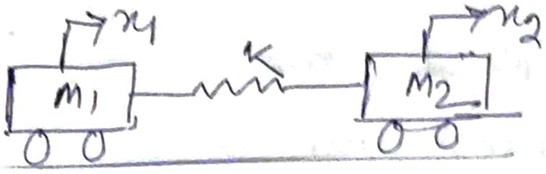
Rearranging the above equations

$$\begin{bmatrix} m & me \\ me & I_0 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 l_4 - k_1 l_3 \\ -k_1 l_3 + k_2 l_4 & k_1 l_3^2 + k_2 l_4^2 \end{bmatrix}\begin{Bmatrix} x_c \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The coupling equation contains static as well as dynamic coupling terms.

If $k_1 l_3 = k_2 l_4$, the system introduces dynamic coupling.

Semi definite System



Consider the masses connected by spring.

$$m_1 \ddot{x}_1 + K(x_1 - x_2) = 0$$

$$m_2 \ddot{x}_2 - K(x_1 - x_2) = 0$$

Let $x_1 = A_1 \sin \omega_n t$, $x_2 = A_2 \sin \omega_n t$, then the above equations reduces to

$$(K - m_1 \omega_n^2) A_1 - K A_2 = 0$$

$$-K A_2 + (K - m_2 \omega_n^2) A_2 = 0$$

For nontrivial solution

$$\begin{vmatrix} K - m_1 \omega_n^2 & -K \\ -K & K - m_2 \omega_n^2 \end{vmatrix} = 0$$

or $m_1 m_2 \omega_n^4 - K(m_1 + m_2)\omega_n^2 = 0$

or $m_1 m_2 \omega_n^2 - K(m_1 + m_2) = 0$

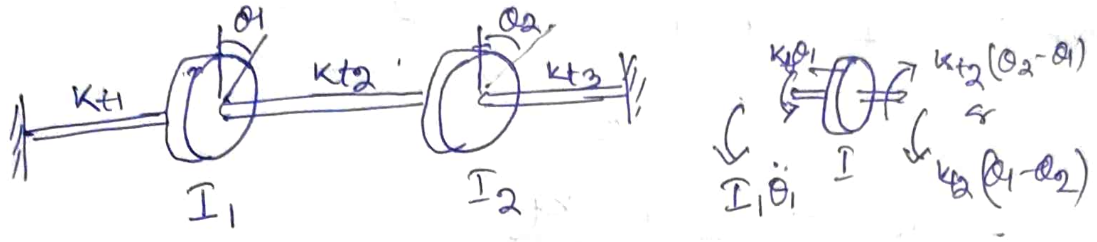
$$\omega_{n_1} = 0 \qquad \omega_{n_2} = \sqrt{\frac{K(m_1 + m_2)}{m_1 m_2}}$$

It shows that one of the natural frequency of the system is zero, which means the system is non-oscillatory, such system are called semi definite system.

Rotational system

Find the natural frequencies and mode shapes for given torsional system.

$$I_1 = I,\ I_2 = 2I,\ K_{t_1} = K_{t_2} = K_{t_3} = K_t$$



The equation of motion

$$I_1\ddot{\theta}_1 + K_t\theta_1 + K_t(\theta_1-\theta_2) = 0$$

$$I_1\ddot{\theta}_1 + 2K_t\theta_1 - K_t\theta_2 = 0 \quad \text{--- (1)}$$

$$I_2\ddot{\theta}_2 - K_t(\theta_1-\theta_2) + K_t\theta_2 = 0$$

$$2I\ddot{\theta}_2 + 2K_t\theta_2 - K_t\theta_1 = 0 \quad \text{--- (2)}$$

Let's assume the solution of equations (1) & (2)

$$\theta_1 = A_1 \sin\omega_n t \quad , \quad \theta_2 = A_2 \sin\omega_n t$$

Then the equation (1) & (2) can be written as

$$(-\omega_n^2 I + 2K_t)A_1 - K_t A_2 = 0 \quad \text{--- 3}$$

$$(-2\omega_n^2 I + 2K_t)A_2 - K_t A_1 = 0 \quad \text{--- (4)}$$

$$\frac{A_1}{A_2} = \frac{K_t}{(-\omega^2 I + 2K_t)} \cancel{\text{or}} \frac{(-2\omega_n^2 I + 2K_t)}{K_t}$$

The frequency can be written as

$$(-2\omega_n^2 I + 2K_t)(-\omega^2 I + 2K_t) - K_t^2 = 0$$

or $2[\omega_n^4 I - \omega_n^2 I K_t - 2K_t\omega_n^2 I + 2K_t^2) - K_t^2 = 0$

or $2\omega_n^4 I^2 - 6k_t I \omega_n^2 + 3k_t^2 = 0$

or $\omega_n^4 - \frac{3k_t}{I}\omega_n^2 + \frac{3k_t^2}{2I^2} = 0$

$$\omega_n^2 = \frac{\frac{3k_t}{I} \pm \sqrt{\left(\frac{3k_t}{I}\right)^2 - 4\cdot\frac{3}{2}\frac{k_t^2}{I^2}}}{2}$$

$$= 1.5\frac{k_t}{I} \pm \frac{\sqrt{3}}{2}\frac{k_t}{I}$$

or $\omega_1 = 1.5\sqrt{\frac{k_t}{I}}$ $\quad \omega_2 = 0.8\sqrt{\frac{k_t}{I}}$

The amplitude ratios are given by

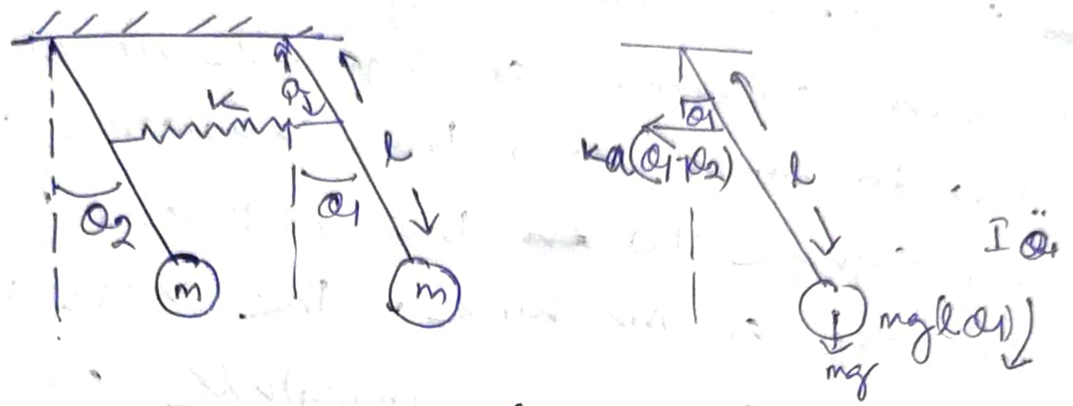
$$\left(\frac{A_1}{A_2}\right)_{\omega_1} = \frac{k_t}{-\left(1.9\sqrt{\frac{k_t}{I}}\right)^2 I + 2k_t} = -4$$

$$\left(\frac{A_1}{A_2}\right)_{\omega_2} = \frac{-2(0.8)^2\frac{k_t}{\cancel{I}}\times\cancel{I} + 2k_t}{k_t} = \frac{-2\times0.64+2}{1}$$

$$= 0.72$$

Transverse vibration

Here two pendulums are coupled by means of a weak spring k, which is unstrained when two pendulum rods are in vertical position.



Assuming ccw angular displacement to be positive and taking moment about the point of suspension, the equation of motions

$$ml^2\ddot{\theta}_1 + mgl\theta_1 + ka^2(\theta_1-\theta_2) = 0$$

$$ml^2\ddot{\theta}_2 + mgl\theta_2 - ka^2(\theta_1-\theta_2) = 0$$

Assuming the normal mode solution

$$\theta_1 = A_1 \sin \omega nt$$

$$\theta_2 = A_2 \sin \omega nt$$

and solve for frequencies

$$\omega_{n1} = \sqrt{g/l} \qquad \omega_{n2} = \sqrt{g/l + 2\frac{ka^2}{ml^2}}$$

$$\left.\frac{A_1}{A_2}\right]_{\omega_{n1}} = 1 \quad , \quad \left.\frac{A_1}{A_2}\right]_{\omega_{n2}} = -1.0$$

Thus in 1st mode, the two pendulum move in phase and spring remains unstretched.

In 2nd mode, the two pendulums move opposite with node at and coupling spring actively involved

Vibration Absorber

When a structure externally excited has undesirable vibrations, it becomes necessary to eliminate them by coupling some vibrating system to it. The vibrating system is known as vibration absorber or dynamic vibration absorber. In such cases the excitation frequency is nearly equal to ~~an~~ the natural frequency of the structure or machine. The mass which is excited can have zero amplitude of vibration and spring mass system (Absorber) which is coupled to it vibrates freely. Vibration absorbers are used to control structural resonance.